Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

A Theory of Geometrical Relations.—*Continued.*[*]

By Arthur Richard Schweitzer.

Chapter VI.

On the Foundations of Grassmann's Extensive Algebra.

(I) Preliminary Remarks on the Genesis of Euclidean Geometry.

In Chapter V we have shown that by the introduction of suitable definitions the system ${}^3K_3$ (and therefore the system ${}^3R_3$) implies a "geometry of position" which is defined by Hilbert's axioms I and II of his "Foundations of Geometry." This geometry may also be defined by Pasch's axioms in §§ 1–2 of his "Vorlesungen über neuere Geometrie" or Peano's axioms I–XVI in his "I principii di Geometria."[†]

The system ${}^3K_3$ may be made *complete*[‡] for euclidean geometry in a manner that is purely descriptive.[§] The procedure necessary for such completion in the case of the "geometries of position" due to Pasch and Peano has been admirably sketched by Burali-Forti[||] and is easily adapted to the system ${}^3K_3$. Burali-Forti says:

"Le système de postulats de M. Pasch, ou celui de M. Peano, donnent toute la Géométrie de position. . . . . Pour obtenir la théorie des parallèles, il faut joindre aux postulats précédentes celui de la continuité due à M. Dedekind et qu'on peut exprimer[¶] en disant que 'Un ensemble convexe de points, contenu sur un segment, est lui-même un segment.'

. . . .

---

[*] This article is a continuation of one under the same title in Vol. XXXI (1909), pp. 365–410; reference to the latter article will be made by pages under the notation "T. G. R."

[†] Compare also, G. Peano, "Sui fondamenti della Geometria," *Rivista di Matematica*, Vol. IV (1894), p. 51; A. B. Kempe, *Proceedings of the London Math. Soc.*, Vol. XXI (1890), pp. 176–179; F. Enriques, *Encyklopaedie der Math. Wiss.*, Band III, Heft 1, pp. 21–27, 70–72.

[‡] Cf. F. Riesz, "Die Genesis des Raumbegriffs," *Math. und Naturw. Berichte aus Ungarn* (1905–1906), p. 309.

[§] Cf. A. B. Kempe, *l. c.*; F. Enriques, *l. c.*, p. 34; A. R. Schweitzer, *Bull. of the Am. Math. Soc.*, Vol. XV (1909), pp. 314–315.

[||] "Verhandlungen des Ersten Internationalen Mathematiker-Kongresses," Leipzig, 1898, pp. 248–249.

[¶] G. Peano, "I principii," pp. 9, 10, 28, 39.

"La definition de demi-droites parallèles donnée[*] on demontre que deux demi-droites parallèles sont complanaires; qu'un point externe à la demi-droite *a* est l'origine d'une seule demi-droite parallèle à *a*; que la relation exprimée par le mot parallèle est réflexive, symétrique et transitive..... Il en résulte aussi que: par un point extérieur à une droite on peut mener une ou deux droites parallèles à la droite même, et qu'un de ces deux cas doit toujours se vérifier. Pour obtenir la Géométrie d'Euclide il faut poser le postulat suivant: 'Quelle que soit la droite *a* il existe un point *b* extérieur à la droite *a* par lequel passe une seule parallèle à la droite *a*'; car on déduit aisément qu'une seule parallèle à la droite *a* passe aussi par un point quelconque."

For further details of the development corresponding to the preceding position we refer to Burali-Forti, *l. c.*; E. H. Moore, *Science*, New Series, Vol. XVII, p. 403,[†] and finally the "Programm" of F. Klein, Erlangen, 1872, reprinted in the *Mathematische Annalen*, Vol. XLIII (1893).[‡]

To complete, therefore, the system ${}^3K_3$ for euclidean geometry, we must first add to this system an axiom of Dedekind continuity. The latter may be expressed in a variety of ways;[§] a form attractive for our purposes is an axiom with the (geometric) content of the so-called Heine-Borel theorem,[||] stated for three dimensions. Finally, a parallel axiom corresponding to that given by Burali-Forti is easily stated. In connection with this axiom it is desirable to give the following definitions:[¶]

1. *The line $\delta\xi'$ does not intersect the* (*coplanar*) *line $a\beta$* means, $\alpha\beta\delta\xi' K_2$ and the existence of ξ, ε such that $\delta\xi'\xi\Delta K_1$ and $\alpha\beta\delta\varepsilon K$ implies $\alpha\beta\delta\varepsilon K\alpha\beta\xi\varepsilon$.

2. *The line $\delta\xi'$ does not intersect the plane $\alpha\beta\gamma$* means, $\alpha\beta\gamma\delta K$ and the existence of ξ such that $\delta\xi'\xi\Delta K_1$ implies $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$.

3. *The plane $\delta\xi'\xi''$ does not intersect the plane $\alpha\beta\gamma$* means, $\alpha\beta\gamma\delta K$, $\delta\xi'\xi''\alpha K$ and the existence of ξ such that $\delta\xi'\xi''\xi K_2$ implies $\alpha\beta\gamma\delta K\alpha\beta\gamma\xi$.

---

[*] G. Peano, "I principii," pp. 38, 39.

[†] Cf. O. Veblen, *Trans. Am. Math. Soc.*, Vol. V (1904), pp. 343–384.

[‡] Cf. Fano, *Encyklopaedie der Math. Wiss.*, Band III, Heft 2, p. 295, etc., and the remarks of the editors of Grassmann's *Gesammelte Werke*, Band I, 1, p. 406, and Band I, 2, pp. 434, 436, 438.

[§] See, for instance, O. Veblen, *l. c.*, pp. 346, 348; F. Riesz, *l. c.*, pp. 318–319; "Atti del IV Congresso Internazionale dei Matematici," Vol. II (1909), pp. 18–24; R. L. Moore, *Trans. Am. Math. Soc.*, Vol. IX (1908), pp. 488–489.

[||] Cf. E. Borel, *Journal de Mathématiques Pures et Appliquées*, 1903, p. 329; *Comptes Rendus*, 1905, p. 298; O. Veblen, *Bull. of the Am. Math. Soc.*, 1904, p. 436.

[¶] For the definition of the symbols used, see "T. G. R." pp. 375–376, 398.

(II) METRICAL COMPLETION[*] OF THE SYSTEM $^3K_3$ FOR EUCLIDEAN GEOMETRY.
EXPLICIT INTRODUCTION OF THE NUMBER SYSTEM.
NUMERICAL DERIVATION OF POINTS.[†]

Although the above observations are important in showing that the outer product of Grassmann may serve as a generating relation for euclidean geometry even when divested very largely of its metrical connotations, yet in our opinion this treatment can not be regarded as conforming to the basal geometric position of Grassmann in his "Ausdehnungslehre" of 1844. Namely, we assume that this position finds its most suitable expression through the use of the outer product (as an indefinable) in connection with the early, fundamental introduction of the number system and the deducing of properties by means of the "numerical derivation" of geometric elements.[‡]

In the following sections we attempt, therefore, a more properly grassmannian development of euclidean geometry.

§ 1. *Quotient Relations: The Classes* $\mathfrak{K}$, $\breve{\mathfrak{K}}$, $\bar{\mathfrak{K}}$.

Associated with the system $^3K_3$[§] is the system $^3\breve{K}_3$ or *conjugate*[||] of $^3K_3$. The latter system is phrased exclusively in terms of the conjugate relation $\breve{K}$ defined as follows: $\alpha\beta\gamma\delta\breve{K}\xi\eta\zeta\tau$ if, and only if, $\alpha\beta\gamma\delta K\eta\xi\zeta\tau$. Thus the validity of the system $^3K_3$ implies the validity of $^3\breve{K}_3$ and conversely.[¶]

More generally, on the basis of the system $^3K_3$, we define two tetrads $\alpha\beta\gamma\delta$, $\xi\eta\zeta\tau$ to be in the *quotient relation*, written $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$, if, and only if, $\alpha\beta\gamma\delta K\xi\eta\zeta\tau$ or $\alpha\beta\gamma\delta\breve{K}\xi\eta\zeta\tau$ or ($\alpha\beta\gamma\delta\bar{K}$ and $\xi\eta\zeta\tau K$); corresponding to the terms of the latter disjunction, the relation $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is said to be in the class $\mathfrak{K}$ or $\breve{\mathfrak{K}}$ or $\bar{\mathfrak{K}}$. The tetradic terms of the preceding quotient relation give rise, through our subsequent developments, to *symmetric*

---

[*] As is easily recognized, this completion differs from that under (I) in *relativity*; the latter completion may be said to be *less absolute* than the former.

[†] In this part we work out, in more exact detail, a position we have briefly indicated in the *Bull. of the Am. Math. Soc.*, Vol. XV (1908), p. 81.

[‡] Cf. Grassmann, *Werke*, Vol. I, 1, p. 10, paragraph 1; p. 138, note 1; F. Caspary, *Bull. des Sciences Math.*, Vol. XIII (1889), pp. 203, 240; G. Peano, *Atti di Torino*, Vol. XXXI (1895–1896), pp. 952–975; Vol. XXXIII (1897–1898), p. 534, concluding remark.

[§] Cf. "T. G. R.," pp. 395–396.

[||] With this term compare the "obverse element" of A. B. Kempe, *l. c.*, p. 155; J. Royce, *Trans. Am. Math. Soc.*, Vol. VI (1905), pp. 357, 365; and the "semi-space" of Laguerre, *Oeuvres*, Tome II (1905), pp. 592–604 and 620, note. See also, "T. G. R.", p. 376, note.

[¶] Use is here made of a logical postulate analogous to that of Russell, "Principles of Mathematics," p. 25, paragraph 2, and p. 97, § 95, (2). Cf. also "T. G. R.," p. 375.

products, as distinguished from *alternating* or *outer* products, which we define below.[*] Grassmann[†] has used a relation $\frac{A_1}{A}$ analogous to our quotient relation in the statement

$$A_1 = \frac{A_1}{A} \cdot A,$$

where A and A_1 are elementary (spatial) magnitudes which are "gleichartig" ($A \neq 0$). From our viewpoint, Grassmann's relation is associated primarily with the system ${}^3R_3$ rather than ${}^3K_3$.[‡]

Axioms 1–16 of the system ${}^3K_3$, as phrased in terms of quotient relations, we shall denote by the notations G_1–G_{16} respectively. By G_{17} we denote correspondingly an axiom which ensures that the system ${}^3K_3$ possesses, on the basis of G_1–G_{16}, the property of Dedekind continuity; suitable forms for this axiom are easily derived from those cited under (I).

§ 2. *Identification of the Classes $\mathfrak{K}$, $\breve{\mathfrak{K}}$, $\bar{\mathfrak{K}}$ with Systems of Real Numbers.*

The classes $\mathfrak{K}$, $\breve{\mathfrak{K}}$, $\bar{\mathfrak{K}}$ are respectively identified with systems of real numbers by means of axioms which we denote by G_{18_1}, G_{18_2}, G_{18_3}. Throughout these axioms we shall denote by "$\theta_1\theta_2\theta_3\theta_4/\theta_1'\theta_2'\theta_3'\theta_4' = q$" that the real number q *corresponds uniquely*[§] to the quotient relation $\theta_1\theta_2\theta_3\theta_4/\theta_1'\theta_2'\theta_3'\theta_4'$.

Axiom G_{18_1}.

1. If the quotient relations[||] $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0$, $\theta_2\theta_3\theta_1\theta_4/\alpha_0\beta_0\gamma_0\delta_0$, $\theta_3\theta_4\theta_1\theta_2/\alpha_0\beta_0\gamma_0\delta_0$ are in the class $\mathfrak{K}$, then $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0 = k^2$, $\theta_2\theta_3\theta_1\theta_4/\alpha_0\beta_0\gamma_0\delta_0 = k^2$, $\theta_3\theta_4\theta_1\theta_2/\alpha_0\beta_0\gamma_0\delta_0 = k^2$.

---

[*] Cf. "T. G. R.," pp. 376, 394.

[†] *Gesammelte Werke*, Vol. I, 1, pp. 41 (§ 8), 123, 127–129, 163 (§ 97), 303.

[‡] Developments similar to those given in this and the subsequent sections might be based directly on the system ${}^3R_3$; cf. "T. G. R.," p. 377, note.

[§] For the purpose of this correspondence the quotient relation is particularized in so far as it is conceived to hold between specific tetrads; to the effective relation in its generality, *i. e.*, to the class of particularized relations, corresponds a class consisting of some or all real numbers; compare B. Russell, "The Principles of Mathematics," pp. 161, 432–433; Grassmann, *Werke*, Vol. I, 1, p. 123, note; p. 138, note 1; pp. 343–345; G. Peano, *Formulaire Mathématique* (1902–03), p. 338: "$u/\psi \varepsilon q$"; p. 260: "$u \times v \varepsilon q$." Grassmann's position (with which Peano seems in harmony) is presumably that the quotients of cospatial magnitudes of the same grade (*i. e.*, "gleichartige Grössen") obey the laws that govern any system of some or all real numbers (cf. *Werke*, Vol. I, 1, p. 134, note), and in accordance with this position it would seem desirable for us to subject our quotient relations postulationally to such laws.

[||] By the notation $\alpha_0\beta_0\gamma_0\delta_0$ we designate in axioms G_{18_1}, G_{18_2}, G_{18_3} the same fixed tetrad (in the K relation to itself); this tetrad may conveniently be identified with the tetrad of axiom 2 of the system ${}^3K_3$, Chapter V, p. 395. The symbol k^2 denotes a positive number.

2. If $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha_0\beta_0\gamma_0\delta_0/\theta_1\theta_2\theta_3\theta_4$ are in the class $\mathfrak{K}$, then $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0 = k^2$ implies $\alpha_0\beta_0\gamma_0\delta_0/\theta_1\theta_2\theta_3\theta_4 = 1/k^2$.

3. If $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha_0\beta_0\gamma_0\delta_0/\theta'_1\theta'_2\theta'_3\theta'_4$ and $\theta_1\theta_2\theta_3\theta_4/\theta'_1\theta'_2\theta'_3\theta'_4$ are in the class $\mathfrak{K}$, then $\theta_1\theta_2\theta_3\theta_4/\alpha_0\beta_0\gamma_0\delta_0 = k^2$ and $\alpha_0\beta_0\gamma_0\delta_0/\theta'_1\theta'_2\theta'_3\theta'_4 = 1/k'^2$ imply $\theta_1\theta_2\theta_3\theta_4/\theta'_1\theta'_2\theta'_3\theta'_4 = k^2/k'^2$.

Axiom G_{18_2}.

1. If $\beta\alpha\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{K}$, then $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$ implies $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$.

2. If $\alpha\beta\gamma\delta/\eta\xi\zeta\tau$ is in the class $\mathfrak{K}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$ implies $\alpha\beta\gamma\delta/\eta\xi\zeta\tau = k^2$.

Axiom G_{18_3}.

1. If $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ is in the class $\bar{\mathfrak{K}}$, then $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = 0$.

2. If $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ is in the class $\bar{\mathfrak{K}}$ and $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau$ is in the class $\mathfrak{K}$ or $\breve{\mathfrak{K}}$, then $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = 0$ and $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau = k^2$ or $-k^2$ imply $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = 0$.

The classes $\mathfrak{K}$, $\breve{\mathfrak{K}}$, $\bar{\mathfrak{K}}$ as satisfying axioms G_{18_1}–G_{18_3} we denote by $\mathfrak{K}_+$, $\mathfrak{K}_-$, $\mathfrak{K}_0$ respectively. We proceed to prove the following theorems:

Theorem 1. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{K}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$ implies $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = k^2$.

For by G_{18_2}, 1, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$ implies $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = -k^2$; therefore by G_{18_2}, 2, $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = k^2$.

Theorem 2. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{K}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$ implies $\alpha\beta\gamma\delta/\eta\xi\zeta\tau = -k^2$.

By theorem 1, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$ implies $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = k^2$; therefore by G_{18_2}, 1, $\alpha\beta\gamma\delta/\eta\xi\zeta\tau = -k^2$.

Theorem 3. If $\beta\alpha\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{K}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$ implies $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$.

By G_{18_2}, 2, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$ implies $\alpha\beta\gamma\delta/\eta\xi\zeta\tau = k^2$; hence by theorem 1, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$.

Theorem 4. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{K}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$.

By hypothesis, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta$ and $\xi\eta\zeta\tau/\xi\eta\zeta\tau$ are both in the class $\mathfrak{K}$. Therefore, by axiom G_{16}, ($\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ or $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$) and ($\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ or $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$) are in $\mathfrak{K}$. Four cases arise.

CASE I. $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ are in $\mathfrak{K}$.

Hence by axiom G_{18_1}, 1, we may put $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k_1^2$ and $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0 = k_2^2$. But if $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ is in $\mathfrak{K}$, so also is $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau$, and by axiom G_{18_1}, 2, $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau = 1/k_2^2$. Hence by G_{18_1}, 3, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k_1^2/k_2^2$, which verifies the theorem.

CASE II. $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ is in $\mathfrak{K}$ and $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ is in $\mathfrak{K}$.

Then $\alpha\beta\gamma\delta/\eta\xi\zeta\tau$ is in $\mathfrak{K}$ and this contradicts the hypothesis.

CASE III. $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ are in $\mathfrak{K}$.

Then, as under Case I, we show that $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = k^2$. Therefore, by theorem 1, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$.

CASE IV. $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ is in $\mathfrak{K}$ and $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ is in $\mathfrak{K}$.

Therefore $\beta\alpha\gamma\delta/\xi\eta\zeta\tau$ is in $\mathfrak{K}$, which contradicts the hypothesis.

Theorem 5. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\breve{\mathfrak{K}}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$.

Since $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in $\breve{\mathfrak{K}}$, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau$ is in $\mathfrak{K}$ (cf. § 1). Hence by theorem 4, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$. Therefore by axiom G_{18_2}, 1, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$.

Theorem 6. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\bar{\mathfrak{K}}$, then $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = 0$.

Since $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\bar{\mathfrak{K}}$, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta$ is not in the class $\mathfrak{K}$, but $\xi\eta\zeta\tau/\xi\eta\zeta\tau$ is in the class $\mathfrak{K}$. Hence, by axiom G_{16}, $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ or $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ is in the class $\mathfrak{K}$; that is, $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau$ or $\alpha_0\beta_0\gamma_0\delta_0/\eta\xi\zeta\tau$ is in the class $\mathfrak{K}$. If $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau$ is in $\mathfrak{K}$, then, by theorem 4, $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau = k^2$. If $\alpha_0\beta_0\gamma_0\delta_0/\eta\xi\zeta\tau$ is in $\mathfrak{K}$, then $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau$ is in $\breve{\mathfrak{K}}$ and, by theorem 5, $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau = -k^2$. Finally, by axiom G_{18_3}, 1, $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = 0$. Thus the hypotheses of axiom G_{18_3}, 2, are fulfilled and we have, therefore, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = 0$.

Theorem 7. The quotient relation $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{K}$, or $\breve{\mathfrak{K}}$, or $\bar{\mathfrak{K}}$ according as $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$, or $-k^2$, or 0.

Suppose that $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$. Then, by definition, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in one of the classes $\mathfrak{K}$, $\breve{\mathfrak{K}}$, $\bar{\mathfrak{K}}$. Now $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ can not be in the class $\breve{\mathfrak{K}}$; for if so, by theorem 5, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$. Similarly, by theorem 6, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ can not be in the class $\bar{\mathfrak{K}}$. Hence $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\mathfrak{K}$.

The remaining cases are proved in an analogous manner.

Theorem 8. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$, then $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = 1/k^2$.

As in theorem 4 we find two essential cases.

CASE I. $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k_1^2$, $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0 = k_2^2$.

Therefore by G_{18_1}, 2, $\alpha_0\beta_0\gamma_0\delta_0/\alpha\beta\gamma\delta = 1/k_1^2$, $\alpha_0\beta_0\gamma_0\delta_0/\xi\eta\zeta\tau = 1/k_2^2$. Hence by G_{18_1}, 3, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k_1^2/k_2^2$, $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = k_2^2/k_1^2$. Thus, since we may take $k^2 = k_1^2/k_2^2$, the theorem is verified.

CASE II. $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k_1^2$, $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0 = k_2^2$.

Then as under Case I we show that $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = k_1^2/k_2^2$, $\eta\xi\zeta\tau/\beta\alpha\gamma\delta = k_2^2/k_1^2$. That is, by theorem 1, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k_1^2/k_2^2$, $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = k_2^2/k_1^2$, and thus the theorem is verified.

Theorem 9. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$ and $\xi\eta\zeta\tau/\alpha'\beta'\gamma'\delta' = 1/k'^2$, then $\alpha\beta\gamma\delta/\alpha'\beta'\gamma'\delta' = k^2/k'^2$.

We have either ($\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$) or ($\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$) in $\mathfrak{K}$. Similarly, ($\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha'\beta'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0$) or ($\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ and $\beta'\alpha'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0$) are in $\mathfrak{K}$. Hence ($\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha'\beta'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0$) or ($\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ and $\eta\xi\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0$ and $\beta'\alpha'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0$) are in $\mathfrak{K}$.

In the former case we may put, by G_{18_1}, 1, $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = l^2$, $\alpha'\beta'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0 = l'^2$, $\xi\eta\zeta\tau/\alpha_0\beta_0\gamma_0\delta_0 = m^2$. Consequently, by G_{18_1}, 2, 3, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = l^2/m^2$, $\xi\eta\zeta\tau/\alpha'\beta'\gamma'\delta' = m^2/l'^2$, $\alpha\beta\gamma\delta/\alpha'\beta'\gamma'\delta' = l^2/l'^2$. Since we may take $l^2/m^2 = k^2$, $m^2/l'^2 = 1/k'^2$ and therefore $l^2/l'^2 = k^2/k'^2$, the theorem is verified.

The remaining case is proved, through theorem 1, in a similar manner.

Theorem 10. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$, then $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta = 1$.*

By theorem 8, if $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$, then $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = 1/k^2$. Hence by theorem 9, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta = 1$.

On the preceding proof we refer to Peano, *Atti di Torino*, Vol. XXXIII, p. 513, axioms 1–3; *Formulaire Mathématique* (1903), p. 3, § 1, and p. 315; Russell, "The Principles of Mathematics," p. 219, § 209, and p. 163.

Theorem 11. If $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta = 1$, then $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = 1$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = 1$.

By axioms G_{16} and G_{18_1}, 1, we have $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0$ or $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$. In the former case, by G_{18_1}, 1, $\beta\gamma\alpha\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$ and $\gamma\delta\alpha\beta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$. Hence by G_{18_1}, 2, 3, $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = 1$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = 1$. In the latter case we have by G_{18_1}, 1, $\gamma\beta\alpha\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$ and $\delta\gamma\alpha\beta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$, and therefore, by G_{18_1}, 2, 3, $\gamma\beta\alpha\delta/\beta\alpha\gamma\delta = 1$ and $\delta\gamma\alpha\beta/\beta\alpha\gamma\delta = 1$; that is, by theorem 1, $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = 1$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = 1$.

Theorem 12. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = k^2$, then $\beta\gamma\alpha\delta/\xi\eta\zeta\tau = k^2$ and $\gamma\delta\alpha\beta/\xi\eta\zeta\tau = k^2$.

By theorem 4, since $\beta\gamma\alpha\delta/\xi\eta\zeta\tau$ and $\gamma\delta\alpha\beta/\xi\eta\zeta\tau$ are in the class $\mathfrak{K}$, $\beta\gamma\alpha\delta/\xi\eta\zeta\tau = l^2$ and $\gamma\delta\alpha\beta/\xi\eta\zeta\tau = m^2$. Hence by theorem 9, $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = l^2/k^2$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = m^2/k^2$. But by theorem 10, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta = 1$, and by theorem 11, $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = 1$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = 1$. Therefore, since the correspondence is unique, we have $l^2 = k^2$ and $m^2 = k^2$, and thus the theorem is verified.

---

* If a tetrad is in the quotient relation to itself, then this quotient relation is necessarily in the class $\mathfrak{K}$. For by definition, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta$ is in $\mathfrak{K}$ or $\breve{\mathfrak{K}}$ or $\bar{\mathfrak{K}}$. If $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta$ is in $\bar{\mathfrak{K}}$, then $\alpha\beta\gamma\delta\bar{K}$ and $\alpha\beta\gamma\delta K$. If $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta$ is in $\breve{\mathfrak{K}}$, then $\alpha\beta\gamma\delta K\beta\alpha\gamma\delta$; that is, by G_{11}, $\alpha\beta\gamma\delta K\alpha\beta\gamma\delta$, which is impossible by G_4.

Theorem 13. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$, then $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = -1/k^2$.

Since $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$, by theorem 3, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$; hence by theorem 8, $\xi\eta\zeta\tau/\beta\alpha\gamma\delta = 1/k^2$; that is, by theorem 2, $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = -1/k^2$.

Theorem 14. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$ and $\xi\eta\zeta\tau/\alpha'\beta'\gamma'\delta' = -1/k'^2$, then $\alpha\beta\gamma\delta/\beta'\alpha'\gamma'\delta' = -k^2/k'^2$.

By theorem 3, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$ and by G_{18_2}, 2, $\xi\eta\zeta\tau/\beta'\alpha'\gamma'\delta' = 1/k'^2$. Therefore, by theorem 9, $\beta\alpha\gamma\delta/\beta'\alpha'\gamma'\delta' = k^2/k'^2$; that is, by G_{18_2}, 1, $\alpha\beta\gamma\delta/\beta'\alpha'\gamma'\delta' = -k^2/k'^2$.

Theorem 14'. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$, then $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = -k^2$.

By axiom G_{18_2}, 2, $\alpha\beta\gamma\delta/\eta\xi\zeta\tau = k^2$; hence by theorem 1, $\beta\alpha\gamma\delta/\xi\eta\zeta\tau = k^2$. Therefore, by theorem 2, $\beta\alpha\gamma\delta/\eta\xi\zeta\tau = -k^2$.

Theorem 15. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$, then $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta = -1$.[*]

By theorem 13, $\xi\eta\zeta\tau/\alpha\beta\gamma\delta = -1/k^2$. Hence by theorem 14, $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta = -1$.

Theorem 16. If $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta = -1$, then $\beta\gamma\alpha\delta/\beta\alpha\gamma\delta = -1$ and $\gamma\delta\alpha\beta/\beta\alpha\gamma\delta = -1$.

Since $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta = -1$, we have by axiom G_{18_2}, 2, $\alpha\beta\gamma\delta/\alpha\beta\gamma\delta = 1$, and hence by theorem 11, $\beta\gamma\alpha\delta/\alpha\beta\gamma\delta = 1$ and $\gamma\delta\alpha\beta/\alpha\beta\gamma\delta = 1$. Therefore, by theorem 2, $\beta\gamma\alpha\delta/\beta\alpha\gamma\delta = -1$ and $\gamma\delta\alpha\beta/\beta\alpha\gamma\delta = -1$.

Theorem 17. If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = -k^2$, then $\beta\gamma\alpha\delta/\xi\eta\zeta\tau = -k^2$ and $\gamma\delta\alpha\beta/\xi\eta\zeta\tau = -k^2$.

Compare the proof of theorem 16 and use theorem 12 instead of theorem 11.

* * *

If $\alpha\beta\gamma\delta/\xi\eta\zeta\tau$ is in the class $\bar{\mathfrak{K}}$, then, as we have seen, $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = 0$, and conversely. By the statement $\alpha\beta\gamma\delta = 0$ we mean, there exists $\xi'\eta'\zeta'\tau'$ such that $\alpha\beta\gamma\delta/\xi'\eta'\zeta'\tau' = 0$. Thus $\alpha\beta\gamma\delta\bar{K}$ and $\alpha\beta\gamma\delta = 0$ are equivalent. Therefore, $\alpha\beta\gamma\delta K$ and $\alpha\beta\gamma\delta \neq 0$ are equivalent. Further, $\alpha\beta\gamma \neq 0$ means that there exists a point δ such that $\alpha\beta\gamma\delta \neq 0$; $\alpha\beta\gamma = 0$ means that there exists no such δ; $\alpha\beta \neq 0$ means $\alpha \neq \beta$.[†]

If $\alpha\beta\gamma\delta \neq 0$, then $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$ or $\beta\alpha\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$. We define: $\alpha\beta\gamma\delta > 0$ means, $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = k^2$. $\alpha\beta\gamma\delta < 0$ means, $\alpha\beta\gamma\delta/\alpha_0\beta_0\gamma_0\delta_0 = -k^2$. Thus, if $\alpha\beta\gamma\delta \neq 0$, then $\alpha\beta\gamma\delta > 0$ or $\alpha\beta\gamma\delta < 0$. In particular, $\alpha_0\beta_0\gamma_0\delta_0 > 0$. By a non-vanishing *alternating product* (Peano) or *outer product* (Grassmann) we mean a tetrad which is greater [less] than zero. When $\alpha\beta\gamma\delta \neq 0$ we term the tetrad a non-vanishing *symmetric product.* Since for any two tetrads $\alpha'\beta'\gamma'\delta'$ and $\alpha''\beta''\gamma''\delta''$ we have the relations $\alpha'\beta'\gamma'\delta'/\alpha_0\beta_0\gamma_0\delta_0 = q'$ and $\alpha''\beta''\gamma''\delta''/\alpha_0\beta_0\gamma_0\delta_0 = q''$, $q', q'' \overset{=}{\gtrless} 0$, and we may define $\alpha'\beta'\gamma'\delta' \overset{=}{\gtrless} \alpha''\beta''\gamma''\delta''$ according as $q' \overset{=}{\gtrless} q''$, symmetric products have the *character of magnitudes** with reference to a fixed tetrad.

---

[*] Compare theorem 10 and the corresponding note; we may show analogously that if the statement $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta$ is valid, then $\alpha\beta\gamma\delta/\beta\alpha\gamma\delta$ is necessarily in the class $\bar{\mathfrak{K}}$.

[†] Cf. Grassmann, *Werke*, Vol. I, 1, § 119.

It may, perhaps, be remarked here that Grassmann, in the "Ausdehnungslehre" of 1844, §§ 137–139 (cf. §§ 3–10), assumes that the outer product is *associative.*† The latter property is easily expressed in terms of our notations of Chapter III‡ by representing an outer product, say $\alpha_1\alpha_2\alpha_3\alpha_4$, by reflexive relations, $\alpha_1\alpha_2\alpha_3\alpha_4 R^{1111}_{1234}$, $\alpha_1\alpha_2(\alpha_3\alpha_4) R^{112}_{123}$, etc., and assuming that these relations are identical. The associativity of the outer product does not seem to enter very essentially into the discussion of the present chapter.

§ 3. *Numerical Derivation of Points.*

By the class $\mathfrak{Q}$ we mean the logical product of the classes $\mathfrak{R}_+$, $\mathfrak{R}_-$, $\mathfrak{R}_0$ (that is, the classes $\mathfrak{R}$, $\breve{\mathfrak{R}}$, $\bar{\mathfrak{R}}$ as conditioned by the axioms G_1–G_{17}, G_{18_1}, G_{18_2}, G_{18_3}). The class $\mathfrak{Q}$ consists, then, of *quotient relations* to each of which corresponds *uniquely* a real number which may be either positive, negative, or zero. As heretofore, the real numbers corresponding to elements of the class $\mathfrak{Q}$ will be designated by small letters of the Latin alphabet.

We proceed to deduce an arbitrary point of a descriptive m-space ($m = 1, 2, 3$) as *numerically derived*§ from $m+1$ independent‖ points of this space. For this purpose we further condition the class $\mathfrak{Q}$ by certain axioms, after setting up the following

Definition 1.

$\alpha\beta\gamma\delta = k\,\alpha'\beta'\gamma'\delta'$ means, there exists $\xi\eta\zeta\tau$ such that $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = m$, $\alpha'\beta'\gamma'\delta'/\xi\eta\zeta\tau = m'$, $m = k\,m'$ and k corresponds to some element of the class $\mathfrak{Q}$.

---

* Cf. Grassmann, *Werke*, Vol. I, 1, pp. 161 (§ 96), 303, note; Veronese, "Grundzüge der Geometrie," p. 18, note 2; Russell, "The Principles of Mathematics," pp. 161, 163.

† Cf. Peano, "Formulaire Mathématique," (1903), p. 339.

‡ See "T. G. R.," p. 376; our Chapters II, III should be compared with Grassmann, *Werke*, Vol. I, 1, pp. 22–45.

§ Grassmann, *Werke*, Vol. I, 2, § 1.

‖ Grassmann, *Werke*, Vol. I, 1, § 107.

Thus if $\alpha\beta\gamma\delta/\alpha'\beta'\gamma'\delta' = q$, then $\alpha\beta\gamma\delta = q\;\alpha'\beta'\gamma'\delta'$; if $\alpha'\beta'\gamma'\delta' \neq 0$, then conversely, $\alpha\beta\gamma\delta = q\;\alpha'\beta'\gamma'\delta'$ implies $\alpha\beta\gamma\delta/\alpha'\beta'\gamma'\delta' = q$. Again, if $\alpha\beta\gamma\delta = q\;\alpha'\beta'\gamma'\delta'$ and $\alpha'\beta'\gamma'\delta' = 0$, then $\alpha\beta\gamma\delta = 0$; but if $\alpha\beta\gamma\delta = 0$, then not necessarily $\alpha'\beta'\gamma'\delta = 0$ unless $q \neq 0$. Also, if $\alpha\beta\gamma\delta = q\;\alpha'\beta'\gamma'\delta'$ and $q = 0$, then $\alpha\beta\gamma\delta = 0$. We notice that any tetrad $\alpha_1\beta_1\gamma_1\delta_1$ has the properties $\alpha_1\beta_1\gamma_1\delta_1 = 1\;\alpha_1\beta_1\gamma_1\delta_1$ and $\alpha_1\beta_1\gamma_1\delta_1 = -1\;\beta_1\alpha_1\gamma_1\delta_1$.

Hereafter we shall write instead of $\alpha\beta\gamma\delta = -1\;\alpha'\beta'\gamma'\delta'$ and $\alpha\beta\gamma\delta = 1\;\alpha'\beta'\gamma'\delta'$ the expressions, $\alpha\beta\gamma\delta = -\alpha'\beta'\gamma'\delta'$ and $\alpha\beta\gamma\delta = \alpha'\beta'\gamma'\delta'$, respectively.

As final axioms of this chapter we give

Axioms G_{19}–G_{23}.

G_{19}. If $\alpha\beta\xi = 0$, $\alpha \neq \beta \neq \xi \neq \alpha$, and there exist two points δ_0', δ_0'' such that $\alpha\beta\delta_0'\delta_0'' \neq 0$ and $\xi\beta\delta_0'\delta_0'' = k_0\;\alpha\xi\delta_0'\delta_0''$, then the existence of δ', δ'' such that $\alpha\beta\delta'\delta'' \neq 0$ and $\xi\beta\delta'\delta'' = k\;\alpha\xi\delta'\delta''$ implies $k = k_0$.

G_{20}.* If $\alpha'\beta'\gamma'\delta' = q'\;\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha''\beta''\gamma''\delta'' = q''\;\alpha_0\beta_0\gamma_0\delta_0$ and q', q'', $q' + q'' \neq 0$, then there exists $\xi\eta\zeta\tau$ such that $\xi\eta\zeta\tau = (q' + q'')\;\alpha_0\beta_0\gamma_0\delta_0$.

G_{21}. If $\alpha\beta\xi = 0$, $\alpha \neq \beta \neq \xi \neq \alpha$, then the existence of δ', δ'' such that $\alpha\beta\delta'\delta'' \neq 0$, $\alpha\xi\delta'\delta'' = k_1\;\alpha\beta\delta'\delta''$, $\xi\beta\delta'\delta'' = k_2\;\alpha\beta\delta'\delta''$ (where k_1 and k_2 are constants† with reference to $\delta'\delta''$) and the existence of $\xi\eta\zeta\tau$ such that $\xi\eta\zeta\tau = (k_1 + k_2)\;\alpha\beta\delta'\delta''$ imply $k_1 + k_2 = 1$.

G_{22}. If $q \neq 0$ corresponds to an element of the class $\mathfrak{Q}$, then $\alpha\beta\gamma\delta \neq 0$ implies the existence of $\xi\eta\zeta\tau$ such that $\xi\eta\zeta\tau = q\;\alpha\beta\gamma\delta$.

G_{23}. If the coplanar lines $\alpha\beta$ and $\xi\xi'$ do not intersect,‡ then there exists some point $\xi_0 \neq \xi, \xi'$ such that: $\xi_0\xi\xi' = 0$ and the existence of δ such that $\xi_0\xi\alpha\delta \neq 0$, $\xi_0\xi'\alpha\delta \neq 0$ implies $\xi_0\xi\beta\delta = \xi_0\xi\alpha\delta$ and $\xi_0\xi'\beta\delta = \xi_0\xi'\alpha\delta$.

In connection with the above definition of the relation $\alpha\beta\gamma\delta = k\;\alpha'\beta'\gamma'\delta'$ we give the following

Definition 2.

$\sum_{i=1}^{m} p_i\alpha_i\beta_i\gamma_i\delta_i = \sum_{j=1}^{n} p^{(j)}\alpha^{(j)}\beta^{(j)}\gamma^{(j)}\delta^{(j)}$ means, there exist for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ the tetrads $\xi_i\eta_i\zeta_i\tau_i$ and $\xi^{(j)}\eta^{(j)}\zeta^{(j)}\tau^{(j)}$ such that $\xi_i\eta_i\zeta_i\tau_i = p_i\alpha_i\beta_i\gamma_i\delta_i$, $\xi^{(j)}\eta^{(j)}\zeta^{(j)}\tau^{(j)} = p^{(j)}\alpha^{(j)}\beta^{(j)}\gamma^{(j)}\delta^{(j)}$ and $\xi_i\eta_i\zeta_i\tau_i = q_i\alpha_0\beta_0\gamma_0\delta_0$, $\xi^{(j)}\eta^{(j)}\zeta^{(j)}\tau^{(j)} = q^{(j)}\alpha_0\beta_0\gamma_0\delta_0$ and $\sum_{i=1}^{m} q_i = \sum_{j=1}^{n} q^{(j)}$.

---

* See axioms G_{18_1}, G_{18_3} for the definition of the tetrad $a\;\beta_0\gamma_0\delta_0$ used in G_{20}.

† Cf. axiom G_{19}.

‡ For definition see 1 under (I) of the present chapter.

By means of our axioms it is easily shown that the relation "=" between the two sums in the above definition is linearly transitive and symmetrical; that terms on one side of the relation may be transposed to the other side after change of sign, etc.* Our definitions 1 and 2 should be carefully compared with Grassmann's "Erklärung 1" and "Erklärung 2," *Werke*, Vol. I, 1, pp. 343–345.

Theorems.

Theorem 18. If $\alpha\beta\gamma\delta = k\,\alpha'\beta'\gamma'\delta'$, then for any tetrad $\theta_1\theta_2\theta_3\theta_4$ such that $\alpha\beta\gamma\delta = p\cdot\theta_1\theta_2\theta_3\theta_4$ and $\alpha'\beta'\gamma'\delta' = p'\,\theta_1\theta_2\theta_3\theta_4$ and $\theta_1\theta_2\theta_3\theta_4 \neq 0$, $p = k\cdot p'$.

If $k = 0$, then (see under definition 1) $\alpha\beta\gamma\delta = 0$ and therefore $p = 0$; hence $p = kp'$. If $k \neq 0$ and $\alpha'\beta'\gamma'\delta' = 0$, then $\alpha\beta\gamma\delta = 0$, and therefore $p = 0$, $p' = 0$; hence $p = kp'$. Let now $k \neq 0$, $\alpha'\beta'\gamma'\delta' \neq 0$. Then $\alpha\beta\gamma\delta/\theta_1\theta_2\theta_3\theta_4 = p$, $\alpha'\beta'\gamma'\delta'/\theta_1\theta_2\theta_3\theta_4 = p'$. By hypothesis and definition 1 there exists $\xi\eta\zeta\tau$ such that $\alpha\beta\gamma\delta/\xi\eta\zeta\tau = m$, $\alpha'\beta'\gamma'\delta'/\xi\eta\zeta\tau = m'$ and $m = km'$. Hence by the theorems of § 2, $\xi\eta\zeta\tau/\theta_1\theta_2\theta_3\theta_4 = p/m$, $\xi\eta\zeta\tau/\theta_1\theta_2\theta_3\theta_4 = p'/m'$. Therefore, $p/m = p'/m'$. But $m = km'$. Hence $p = kp'$.

Theorem 18′. If $\alpha\beta\gamma\delta = k\,\alpha'\beta'\gamma'\delta'$ and $k \neq 0$, then $\alpha'\beta'\gamma'\delta' = 1/k\;\alpha\beta\gamma\delta$.

This is easily shown by the theorems of § 2 if $\alpha'\beta'\gamma'\delta' \neq 0$. Suppose that $\alpha'\beta'\gamma'\delta' = 0$; then $\alpha\beta\gamma\delta = 0$, and we must show that $1/k$ corresponds to an element of the class $\mathfrak{Q}$. Since $k \neq 0$ and $\alpha_0\beta_0\gamma_0\delta_0 \neq 0$ (cf. note on axiom G_{20}), there exists by axiom G_{22} the tetrad $\xi\eta\zeta\tau$ such that $\xi\eta\zeta\tau = k\,\alpha_0\beta_0\gamma_0\delta_0$; that is, by the theorems of § 2, $\alpha_0\beta_0\gamma_0\delta_0 = 1/k\;\xi\eta\zeta\tau$. Hence $1/k$ corresponds to an element of the class $\mathfrak{Q}$.

Theorem 18″. If $\alpha\beta\gamma\delta = k\,\alpha'\beta'\gamma'\delta'$ and $\alpha'\beta'\gamma'\delta' = k'\,\alpha''\beta''\gamma''\delta''$, then $\alpha\beta\gamma\delta = kk'\,\alpha''\beta''\gamma''\delta''$.

If $\alpha'\beta'\gamma'\delta' = 0$, then the theorem is easily verified when $k = 0$, or when $k \neq 0$ and $k' = 0$. Let now $\alpha'\beta'\gamma'\delta' = 0$ and $k, k' \neq 0$; then $\alpha''\beta''\gamma''\delta'' = 0$, $\alpha\beta\gamma\delta = 0$, and we must show that kk' corresponds to an element of the class $\mathfrak{Q}$. Since $\alpha_0\beta_0\gamma_0\delta_0 \neq 0$ there exist by axiom G_{22} the tetrads $\xi_1\eta_1\zeta_1\tau_1$ and $\xi_2\eta_2\zeta_2\tau_2$ such that $\xi_1\eta_1\zeta_1\tau_1 = k\,\alpha_0\beta_0\gamma_0\delta_0$ and $\xi_2\eta_2\zeta_2\tau_2 = k'\,\xi_1\eta_1\zeta_1\tau_1$. Hence, by the theorems of § 2, $\xi_2\eta_2\zeta_2\tau_2 = kk'\,\alpha_0\beta_0\gamma_0\delta_0$, and therefore kk' corresponds to an element of the class $\mathfrak{Q}$.

In the case $\alpha'\beta'\gamma'\delta' \neq 0$, $k, k' \neq 0$ the theorem is verified by referring to the theorems of § 2. Finally, when $\alpha'\beta'\gamma'\delta' \neq 0$ and $k = 0$, the theorem is obviously valid.

---

* In connection with definition 2, compare axiom G_{20}.

Theorem 19. If $\alpha'\beta'\gamma'\delta' = q'\,\xi\eta\zeta\tau$ and $\alpha''\beta''\gamma''\delta'' = q''\,\xi\eta\zeta\tau$, where $\xi\eta\zeta\tau \neq 0$, then $\alpha'\beta'\gamma'\delta' + \alpha''\beta''\gamma''\delta'' = (q' + q'')\,\xi\eta\zeta\tau$.

We must show that there exists $\xi_1\eta_1\zeta_1\tau_1$ such that $\xi_1\eta_1\zeta_1\tau_1 = (q' + q'')$ $\xi\eta\zeta\delta$ and that if $\alpha'\beta'\gamma'\delta' = p'\,\alpha_0\beta_0\gamma_0\delta_0$, $\alpha''\beta''\gamma''\delta'' = p''\,\alpha_0\beta_0\gamma_0\delta_0$ and $\xi_1\eta_1\zeta_1\tau_1 = p\,\alpha_0\beta_0\gamma_0\delta_0$, then $p' + p'' = p$.

Let $\xi\eta\zeta\tau = p_1\,\alpha_0\beta_0\gamma_0\delta_0$. Then, since by hypothesis $\alpha'\beta'\gamma'\delta' = q'\,\xi\eta\zeta\tau$ and $\alpha''\beta''\gamma''\delta'' = q''\,\xi\eta\zeta\tau$, we have $\alpha'\beta'\gamma'\delta' = p_1 q'\,\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha''\beta''\gamma''\delta'' = p_1 q''\,\alpha_0\beta_0\gamma_0\delta_0$. Hence by G_{20} there exists a tetrad, say $\xi_1\eta_1\zeta_1\tau_1$, such that $\xi_1\eta_1\zeta_1\tau_1 = (p_1 q' + p_1 q'')\,\alpha_0\beta_0\gamma_0\delta_0$. But $\alpha_0\beta_0\gamma_0\delta_0 = \frac{1}{p_1}\xi\eta\zeta\tau$ since $\xi\eta\zeta\tau \neq 0$ and $\alpha_0\beta_0\gamma_0\delta_0 \neq 0$. Hence $\xi_1\eta_1\zeta_1\tau_1 = (q' + q'')\,\xi\eta\zeta\tau$.

From the preceding we evidently have $p' = p_1 q'$, $p'' = p_1 q''$, $p = p_1 q' + p_1 q''$, and hence $p' + p'' = p$.

Theorem 20. If $\alpha\beta\gamma\delta = k\,\xi\eta\zeta\tau$ and $\xi\eta\zeta\tau = \alpha'\beta'\gamma'\delta' + \alpha''\beta''\gamma''\delta''$, then $\alpha\beta\gamma\delta = k\,\alpha'\beta'\gamma'\delta' + k\,\alpha''\beta''\gamma''\delta''$.

If $k = 0$, then $\alpha\beta\gamma\delta = 0$ and the theorem is obviously valid. Therefore we suppose $k \neq 0$. Likewise, we may assume that $\alpha'\beta'\gamma'\delta' \neq 0$ and $\alpha''\beta''\gamma''\delta'' \neq 0$.

Since k corresponds to an element of the class $\mathfrak{Q}$ (see definition 1), by axiom G_{22} there exist, say, $\lambda'\mu'\nu'\omega'$ and $\lambda''\mu''\nu''\omega''$, such that $\lambda'\mu'\nu'\omega' = k\,\alpha'\beta'\gamma'\delta'$ and $\lambda''\mu''\nu''\omega'' = k\,\alpha''\beta''\gamma''\delta''$. Let $\alpha'\beta'\gamma'\delta' = q'\,\alpha_0\beta_0\gamma_0\delta_0$ and $\alpha''\beta''\gamma''\delta'' = q''\,\alpha_0\beta_0\gamma_0\delta_0$. Then $\lambda'\mu'\nu'\omega' = kq'\,\alpha_0\beta_0\gamma_0\delta_0$ and $\lambda''\mu''\nu''\omega'' = kq''\,\alpha_0\beta_0\gamma_0\delta_0$, and we must show that $\alpha\beta\gamma\delta = (kq' + kq'')\,\alpha_0\beta_0\gamma_0\delta_0$. Now by hypothesis $\xi\eta\zeta\tau = \alpha'\beta'\gamma'\delta' + \alpha''\beta''\gamma''\delta''$, and therefore, by definition 2, $\xi\eta\zeta\tau = (q' + q'')\,\alpha_0\beta_a\gamma_0\delta_0$. If $\xi\eta\zeta\tau = 0$, $q' + q'' = 0$, $\alpha\beta\gamma\delta = 0$ and therefore $\alpha\beta\gamma\delta = (kq' + kq'')\,\alpha_0\beta_0\gamma_0\delta_0$. If $\xi\eta\zeta\tau \neq 0$, then from $\alpha\beta\gamma\delta = k\,\xi\eta\zeta\tau$ and $\xi\eta\zeta\tau = (q + q')\,\alpha_0\beta_0\gamma_0\delta_0$ we obtain $\alpha\beta\gamma\delta = (kq + kq')\,\alpha_0\beta_0\gamma_0\delta_0$.

Theorem 21. If $\xi\eta\zeta\tau = (k + k')\,\alpha\beta\gamma\delta$, where k and k' correspond to elements of the class $\mathfrak{Q}$, then $\xi\eta\zeta\tau = k\,\alpha\beta\gamma\delta + k'\,\alpha\beta\gamma\delta$.

We assume $\alpha\beta\gamma\delta \neq 0$. By axiom G_{22} there exist $\lambda\mu\nu\omega$ and $\lambda'\mu'\nu'\omega'$ such that $\lambda\mu\nu\omega = k\,\alpha\beta\gamma\delta$ and $\lambda'\mu'\nu'\omega' = k'\,\alpha\beta\gamma\delta$. Let $\lambda\mu\nu\omega = q\,\alpha_0\beta_0\gamma_0\delta_0$ and $\lambda'\mu'\nu'\omega' = q'\,\alpha_0\beta_0\gamma_0\delta_0$. Then we must show that $\xi\eta\zeta\tau = (q + q')\,\alpha_0\beta_0\gamma_0\delta_0$.

We suppose that $k \neq 0$; then $\alpha\beta\gamma\delta = 1/k\,\lambda\mu\nu\omega$ and since $\lambda\mu\nu\omega = q\,\alpha_0\beta_0\gamma_0\delta_0$ we have, $\alpha\beta\gamma\delta = q/k\,\alpha_0\beta_0\gamma_0\delta_0$. But by hypothesis, $\xi\eta\zeta\tau = (k + k')\,\alpha\beta\gamma\delta$. Hence $\xi\eta\zeta\tau = (q + qk'/k)\,\alpha_0\beta_0\gamma_0\delta_0$. Since $\lambda'\mu'\nu'\omega' = k'\,\alpha\beta\gamma\delta$ and $\alpha\beta\gamma\delta = q/k \cdot \alpha_0\beta_0\gamma_0\delta_0$, we have $\lambda'\mu'\nu'\omega' = qk'/k\,\alpha_0\beta_0\gamma_0\delta_0$. But $\lambda'\mu'\nu'\omega' = q'\,\alpha_0\beta_0\gamma_0\delta_0$. Therefore $q' = qk'/k$; that is, $\xi\eta\zeta\tau = (q + q')\,\alpha_0\beta_0\gamma_0\delta_0$.

Theorem 22. If $p\,\alpha\beta\gamma\delta = q\,\xi\eta\zeta\tau$ and $p'\,\alpha'\beta'\gamma'\delta' = q'\,\xi'\eta'\zeta'\tau'$, then $p\,\alpha\beta\gamma\delta + p'\,\alpha'\beta'\gamma'\delta' = q\,\xi\eta\zeta\tau + q'\,\xi'\eta'\zeta'\tau'$.

By hypothesis and definition there exist tetrads, say, $\lambda'\mu'\nu'\omega'$, $\lambda''\mu''\nu''\omega''$, $\lambda_1\mu_1\nu_1\omega_1$, $\lambda_2\mu_2\nu_2\omega_2$, such that $\lambda'\mu'\nu'\omega' = p\,\alpha\beta\gamma\delta$, $\lambda''\mu''\nu''\omega'' = p'\,\alpha'\beta'\gamma'\delta'$, $\lambda_1\mu_1\nu_1\omega_1 = q\,\xi\eta\zeta\tau$, $\lambda_2\mu_2\nu_2\omega_2 = q'\,\xi'\eta'\zeta'\tau'$. Let $\lambda'\mu'\nu'\omega' = r'\,\alpha_0\beta_0\gamma_0\delta_0$, $\lambda''\mu''\nu''\omega'' = r''\,\alpha_0\beta_0\gamma_0\delta_0$, $\lambda_1\mu_1\nu_1\omega_1 = r_1\alpha_0\beta_0\gamma_0\delta_0$, $\lambda_2\mu_2\nu_2\omega_2 = r_2\alpha_0\beta_0\gamma_0\delta_0$. Then by theorem 19, $\lambda'\mu'\nu'\omega' + \lambda''\mu''\nu''\omega'' = (r' + r'')\,\alpha_0\beta_0\gamma_0\delta_0$ and $\lambda_1\mu_1\nu_1\omega_1 + \lambda_2\mu_2\nu_2\omega_2 = (r_1 + r_2)\,\alpha_0\beta_0\gamma_0\delta_0$. But by hypothesis, $r' = r_1$ and $r'' = r_2$; hence $r' + r'' = r_1 + r_2$, which verifies the theorem.

Theorems 19–22 have obvious extensions to an arbitrary number of terms. These theorems are essentially involved in the proof of the remaining theorems of this section.

In the following we shall denote (cf. Chapter V) by the notations Δ', Δ'', Δ_1, Δ_2, $\Delta^{(1)}$, $\Delta^{(2)}$, etc., *arbitrary points.*

Theorem 23.* If $\alpha\beta\xi = 0$, $\alpha \neq \beta \neq \xi \neq \alpha$, then $\xi\beta\Delta_1\Delta_2 = k\,\alpha\xi\Delta_1\Delta_2$, where k is constant with reference to Δ_1, Δ_2.

Let δ_1, δ_2 be two given points. If $\alpha\beta\delta_1\delta_2 \neq 0$, then by axioms G_{18_1}–G_{18_3}, G_{19} and the theorems of Chapter V we readily show that $\xi\beta\delta_1\delta_2 = k\,\alpha\xi\delta_1\delta_2$.

Suppose now $\alpha\beta\delta_1\delta_2 = 0$. Then we show that $\xi\beta\delta_1\delta_2 = 0$ and $\alpha\xi\delta_1\delta_2 = 0$. If $\alpha\beta\delta_1 \neq 0$, then since $\alpha\beta\xi = 0$, we have by theorem 7, Chapter V, $\xi\beta\delta_1\delta_2 = 0$, $\alpha\xi\delta_1\delta_2 = 0$. If $\alpha\beta\delta_1 = 0$, then since $\alpha\beta\xi = 0$, supposing $\xi \neq \delta_1$, we have by theorem 11, Chapter V, $\xi\beta\delta_1\delta_2 = 0$, $\alpha\xi\delta_1\delta_2 = 0$.

Theorem 24. If $\alpha\beta\xi = 0$, $\alpha \neq \beta \neq \xi \neq \alpha$, then $\alpha\beta\Delta_1\Delta_2 = \alpha\xi\Delta_1\Delta_2 + \xi\beta\Delta_1\Delta_2$.

If $\alpha\beta\xi = 0$ and α, β, ξ are distinct, then by axioms 1 and 2 of the system ${}^3K_3$ and theorems 5, 8 of Chapter V there exist two points δ_0', δ_0'' such that $\alpha\beta\delta_0'\delta_0'' \neq 0$, $\xi\beta\delta_0'\delta_0'' \neq 0$, $\alpha\xi\delta_0'\delta_0'' \neq 0$. Also $\alpha\beta\xi = 0$ implies $\xi\beta\alpha = 0$ and $\alpha\xi\beta = 0$. Thus from axiom G_{19} follow two theorems which are obtained from this axiom by interchanging ξ and α, ξ and β respectively. Hence for any two points δ', δ'' such that $\xi\beta\delta'\delta'' \neq 0$ and $\alpha\xi\delta'\delta'' \neq 0$ we have, $\alpha\beta\delta'\delta'' = k_1\xi\alpha\delta'\delta''$ and $\beta\xi\delta'\delta'' = k_2\alpha\beta\delta'\delta''$, where k_1 and k_2 are constants with reference to $\delta'\delta''$. Then by axiom G_{18_2}, 1, 2, and theorems 2, 8, 13 of § 2, we have $\alpha\xi\delta'\delta'' = -\dfrac{1}{k_1}\alpha\beta\delta'\delta''$ and $\xi\beta\delta'\delta'' = -k_2\alpha\beta\delta'\delta''$. Let now $\alpha\beta\delta'\delta'' =$

---

* In this, and the following theorems, arbitrary points occurring homogeneously in relations may be omitted by introducing a suitable convention.

$l \cdot \alpha_0 \beta_0 \gamma_0 \delta_0$; then by theorems of § 2 we have, $\alpha \xi \delta' \delta'' = -l/k_1\, \alpha_0 \beta_0 \gamma_0 \delta_0$ and $\xi \beta \delta' \delta'' = -l k_2 \alpha_0 \beta_0 \gamma_0 \delta_0$. Hence by axiom G_{20} and definition 2 we have,

$$\alpha \xi \delta' \delta'' + \xi \beta \delta' \delta'' = -\, (l/k_1 + l k_2)\, \alpha_0 \beta_0 \gamma_0 \delta_0;$$

that is, since $\alpha_0 \beta_0 \gamma_0 \delta_0 = 1/l\, \alpha \beta \delta' \delta''$,

$$\alpha \xi \delta' \delta'' + \xi \beta \delta' \delta'' = -\, (1/k_1 + k_2)\, \alpha \beta \delta' \delta'',$$

where k_1 and k_2 are constants with reference to δ', δ''. Accordingly, by axiom G_{21} we have $-(1/k_1 + k_2) = 1$, which establishes the theorem when $\alpha \beta \delta' \delta'' \neq 0$.

If $\alpha \beta \delta' \delta'' = 0$, then, as under the preceding theorem, we show that $\alpha \xi \delta' \delta'' = 0$, $\xi \beta \delta' \delta'' = 0$.

Theorem 25. If the line $\xi \xi'$ does not intersect the coplanar line $\alpha \beta$, then $\alpha \xi \xi' \Delta = \beta \xi \xi' \Delta$ and $\alpha \beta \xi \Delta = \alpha \beta \xi' \Delta$.

We show first that if $\xi \xi'$ does not intersect $\alpha \beta$, then the existence of δ such that $\alpha \xi \xi' \delta \neq 0$ implies $\alpha \xi \xi' \delta = \beta \xi \xi' \delta$. By axiom G_{23} there exists the point $\xi_0 \neq \xi, \xi'$ such that: $\xi_0 \xi \xi' = 0$ and the existence of δ such that $\xi_0 \xi \alpha \delta \neq 0$ and $\xi_0 \xi' \alpha \delta \neq 0$ implies $\xi_0 \xi \alpha \delta = \xi_0 \xi \beta \delta$ and $\xi_0 \xi' \alpha \delta = \xi_0 \xi' \beta \delta$. By definition 1 under (I) of the present chapter, since $\xi \xi'$ does not intersect $\alpha \beta$, $\alpha \xi \xi' \neq 0$. Let the point δ_1 be such that $\alpha \xi \xi' \delta_1 \neq 0$; then since $\xi_0 \xi \xi' = 0$, we have by theorem 8, Chapter V, $\xi_0 \xi \alpha \delta_1 \neq 0$, $\xi_0 \xi' \alpha \delta_1 \neq 0$. Hence by axiom G_{23}, $\xi_0 \xi \alpha \delta_1 = \xi_0 \xi \beta \delta_1$ and $\xi_0 \xi' \alpha \delta_1 = \xi_0 \xi' \beta \delta_1$. Therefore, by theorem 22,

$$\xi \xi_0 \alpha \delta_1 + \xi_0 \xi' \alpha \delta_1 = \xi \xi_0 \beta \delta_1 + \xi_0 \xi' \beta \delta_1 .$$

But by theorem 24, since $\xi_0 \xi \xi' = 0$,

$$\xi \xi_0 \alpha \delta_1 + \xi_0 \xi' \alpha \delta_1 = \xi \xi' \alpha \delta_1 ,$$
$$\xi \xi_0 \beta \delta_1 + \xi_0 \xi' \beta \delta_1 = \xi \xi' \beta \delta_1 .$$

Therefore, by definition 2, $\xi \xi' \alpha \delta_1 = \xi \xi' \beta \delta_1$.

Let δ_2 be such that $\alpha \xi \xi' \delta_2 = 0$; then, since $\alpha \xi \xi' \beta = 0$ and $\alpha \xi \xi' \neq 0$, we have by theorem 7 of Chapter V, $\beta \xi \xi' \delta_2 = 0$.

We have thus verified the first part of the preceding theorem; and since the relation of non-intersection is symmetrical, it is clear that the remaining part of the theorem is proved in a similar manner.

Theorem 26. If $\alpha \beta \gamma \neq 0$ and $\alpha \beta \gamma \xi = 0$, then $\xi \beta \gamma \Delta + \alpha \xi \gamma \Delta + \alpha \beta \xi \Delta = \alpha \beta \gamma \Delta$.

If $\xi = \alpha$ or β or γ, then the theorem obviously is verified.

Let now $\xi \neq \alpha, \beta, \gamma$. Then by the theorems of Chapter V there exists a point η such that ($\beta\eta\xi = 0$ and $\alpha\gamma\eta = 0$) or ($\alpha\eta\xi = 0$ and $\beta\gamma\eta = 0$) or ($\gamma\eta\xi = 0$ and $\alpha\beta\eta = 0$). If $\eta = \alpha$ or β or γ, the theorem is easily verified by theorem 24 and definition 2. Assuming $\eta \neq \alpha, \beta, \gamma$, we have by theorem 24, if $\beta\eta\xi = 0$ and $\alpha\gamma\eta = 0$,

$$\alpha\eta\Delta'\Delta + \eta\gamma\Delta'\Delta = \alpha\gamma\Delta'\Delta, \tag{1}$$

$$\beta\eta\Delta''\Delta + \eta\xi\Delta''\Delta = \beta\xi\Delta''\Delta. \tag{2}$$

Hence, putting in (1) $\Delta' = \xi, \beta$ and in (2) $\Delta'' = \alpha, \gamma$, we obtain respectively,

$$\left.\begin{aligned} \alpha\eta\xi\Delta + \eta\gamma\xi\Delta &= \alpha\gamma\xi\Delta, \\ \alpha\eta\beta\Delta + \eta\gamma\beta\Delta &= \alpha\gamma\beta\Delta, \\ \beta\eta\alpha\Delta + \eta\xi\alpha\Delta &= \beta\xi\alpha\Delta, \\ \beta\eta\gamma\Delta + \eta\xi\gamma\Delta &= \beta\xi\gamma\Delta. \end{aligned}\right\} \tag{3}$$

From the right-hand side of (3) we obtain expressions for $\xi\beta\gamma\Delta$, $\alpha\xi\gamma\Delta$, $\alpha\beta\xi\Delta$, $\alpha\beta\gamma\Delta$; then the theorem is seen to be valid by finding the sum (cf. theorem 22) $\xi\beta\gamma\Delta + \alpha\xi\gamma\Delta + \alpha\beta\xi\Delta$ and comparing with the expression for $\alpha\beta\gamma\Delta$.

Corollary. If $\alpha\beta\gamma = 0$, then for any ξ, $\xi\beta\gamma\Delta + \alpha\xi\gamma\Delta + \alpha\beta\xi\Delta = \alpha\beta\gamma\Delta$.

For if $\xi\beta\gamma$, $\alpha\xi\gamma$, $\alpha\beta\xi$ are all equal to zero, then by axiom 7 of system ${}^3K_3$, Chapter V, $\alpha\beta\gamma = 0$; thus the theorem is verified. Let now $\xi\beta\gamma$ or $\alpha\xi\gamma$ or $\alpha\beta\xi$ be not equal to zero, say $\xi\beta\gamma \neq 0$; then by theorem 26 for the point α we have, since $\xi\beta\gamma\alpha = 0$,

$$\alpha\beta\gamma\Delta + \xi\alpha\gamma\Delta + \xi\beta\alpha\Delta = \xi\beta\gamma\Delta;$$

that is,

$$\xi\beta\gamma\Delta + \alpha\xi\gamma\Delta + \alpha\beta\xi\Delta = \alpha\beta\gamma\Delta.$$

Theorem 27. If $\alpha\beta\gamma\delta \neq 0$, then for any ξ, $\xi\beta\gamma\delta + \alpha\xi\gamma\delta + \alpha\beta\xi\delta + \alpha\beta\gamma\xi = \alpha\beta\gamma\delta$.

If $\xi = \alpha$ or β or γ or δ, then the theorem obviously is verified.

Let $\xi \neq \alpha, \beta, \gamma, \delta$. Then by the theorems of Chapter V there exists a point η such that ($\alpha\eta\xi = 0$ and $\beta\gamma\delta\eta = 0$) or ($\beta\eta\xi = 0$ and $\alpha\gamma\delta\eta = 0$) or ($\gamma\eta\xi = 0$ and $\alpha\beta\delta\eta = 0$) or ($\delta\eta\xi = 0$ and $\alpha\beta\gamma\eta = 0$). If η is collinear with an edge of $\alpha\beta\gamma\delta$, then the theorem is easily verified by theorem 26 and definition 2. Assuming $\alpha\beta\eta \neq 0$, $\beta\gamma\eta \neq 0$, etc., we have by theorem 24, if $\alpha\eta\xi = 0$ and $\beta\gamma\delta\eta = 0$,

$$\alpha\eta\Delta\Delta' + \eta\xi\Delta\Delta' = \alpha\xi\Delta\Delta',$$

and therefore,

$$\left.\begin{aligned} \alpha\eta\beta\gamma + \eta\xi\beta\gamma &= \alpha\xi\beta\gamma, \\ \alpha\eta\gamma\delta + \eta\xi\gamma\delta &= \alpha\xi\gamma\delta, \\ \alpha\eta\beta\delta + \eta\xi\beta\delta &= \alpha\xi\beta\delta. \end{aligned}\right\} \tag{1}$$

Also by theorem 26 we have for $\eta\gamma\delta$ and the point β, since $\beta\gamma\delta\eta = 0$,

$$\beta\gamma\delta\Delta'' + \eta\beta\delta\Delta'' + \eta\gamma\beta\Delta'' = \eta\gamma\delta\Delta'',$$

and consequently for $\Delta'' = \alpha, \xi$ we obtain respectively,

$$\left.\begin{aligned} \beta\gamma\delta\alpha + \eta\beta\delta\alpha + \eta\gamma\beta\alpha &= \eta\gamma\delta\alpha, \\ \beta\gamma\delta\xi + \eta\beta\delta\xi + \eta\gamma\beta\xi &= \eta\gamma\delta\xi. \end{aligned}\right\} \tag{2}$$

From (1) and (2) we obtain expressions for $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$; whence by addition and comparing the sum obtained with the expression for $\alpha\beta\gamma\delta$ in (2), we easily verify the theorem.

Corollary. If $\alpha\beta\gamma\delta = 0$, then for any ξ, $\xi\beta\gamma\delta + \alpha\xi\gamma\delta + \alpha\beta\xi\delta + \alpha\beta\gamma\xi = \alpha\beta\gamma\delta$.

For if $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ are all equal to zero, then by axiom 7 of the system ${}^3K_3$, Chapter V, $\alpha\beta\gamma\delta = 0$ and the theorem is verified. Let now $\xi\beta\gamma\delta$, $\alpha\xi\gamma\delta$, $\alpha\beta\xi\delta$, $\alpha\beta\gamma\xi$ be not all zero, say $\xi\beta\gamma\delta \neq 0$. Then by theorem 27, for the point α we have

$$\alpha\beta\gamma\delta + \xi\alpha\gamma\delta + \xi\beta\alpha\delta + \xi\beta\gamma\alpha = \xi\beta\gamma\delta;$$

that is,

$$\xi\beta\gamma\delta + \alpha\xi\gamma\delta + \alpha\beta\xi\delta + \alpha\beta\gamma\xi = \alpha\beta\gamma\delta.$$

Theorem 28.[*] If $\alpha\beta\xi = 0$ and $\alpha \neq \beta \neq \xi \neq \alpha$,[†] then

$$\xi\Delta_1\Delta_2\Delta_3 = a'\alpha\Delta_1\Delta_2\Delta_3 + b'\beta\Delta_1\Delta_2\Delta_3,$$

where $a' + b' = 1$.

Since $\alpha\beta\xi = 0$ and α, β, ξ are distinct, we have by theorem 23,

$$\xi\beta\Delta'\Delta'' = k'\xi\alpha\Delta'\Delta''. \tag{1}$$

Let $\delta_1, \delta_2, \delta_3$ be three given points and suppose first that $\delta_1\delta_2\delta_3 \neq 0$. Then we distinguish three cases.

---

[*] Compare Peano, "Calcolo Geometrico," pp. 33–35.

[†] If $\xi = \alpha$ we have, by definition 2, $\xi\Delta_1\Delta_2\Delta_3 = 1\,\alpha\Delta_1\Delta_2\Delta_3 + 0\,\beta\Delta_1\Delta_2\Delta_3$, and similarly if $\xi = \beta$.

CASE I. The plane $\delta_1\delta_2\delta_3$ meets the line $\alpha\beta$ in the unique point $\xi^{(0)}$.* Then by theorem 24 we have,

$$\left.\begin{aligned}\xi\beta\Delta'\Delta'' &= \xi\xi^{(0)}\Delta'\Delta'' + \xi^{(0)}\beta\Delta'\Delta'',\\ \xi\alpha\Delta'\Delta'' &= \xi\xi^{(0)}\Delta'\Delta'' + \xi^{(0)}\alpha\Delta'\Delta''.\end{aligned}\right\} \tag{2}$$

Hence,

$$\left.\begin{aligned}\xi\beta\delta_1\delta_2 &= \xi\xi^{(0)}\delta_1\delta_2 + \xi^{(0)}\beta\delta_1\delta_2,\\ \xi\alpha\delta_1\delta_2 &= \xi\xi^{(0)}\delta_1\delta_2 + \xi^{(0)}\alpha\delta_1\delta_2.\end{aligned}\right\} \tag{2'}$$

But from (1),

$$\xi\beta\delta_1\delta_2 = k'\xi\alpha\delta_1\delta_2. \tag{1'}$$

Hence from (1′) and the second relation of (2′), by theorem 20,

$$\xi\beta\delta_1\delta_2 = k'\xi\xi^{(0)}\delta_1\delta_2 + k'\xi^{(0)}\alpha\delta_1\delta_2. \tag{3}$$

Therefore from the first relation of (2′) and from (3),

$$\xi\xi^{(0)}\delta_1\delta_2 + \xi^{(0)}\beta\delta_1\delta_2 = k'\xi\xi^{(0)}\delta_1\delta_2 + k'\xi^{(0)}\alpha\delta_1\delta_2.$$

That is,

$$\xi\xi^{(0)}\delta_1\delta_2 - k'\xi\xi^{(0)}\delta_1\delta_2 = -k'\alpha\xi^{(0)}\delta_1\delta_2 + \beta\xi^{(0)}\delta_1\delta_2.$$

Now by axiom G_{22} there exist $\xi'\eta'\zeta'\tau'$ and $\xi''\eta''\zeta''\tau''$ such that $\xi'\eta'\zeta'\tau' = \xi\xi^{(0)}\delta_1\delta_2$ and $\xi''\eta''\zeta''\tau'' = -k'\xi\xi^{(0)}\delta_1\delta_2$; hence by theorem 19 there exists the sum, say $\xi\eta\zeta\tau$, such that

$$\xi'\eta'\zeta'\tau' + \xi''\eta''\zeta''\tau'' = (1-k')\,\xi\xi^{(0)}\delta_1\delta_2,$$

and therefore,

$$(1-k')\,\xi\xi^{(0)}\delta_1\delta_2 = -k'\alpha\xi^{(0)}\delta_1\delta_2 + \beta\xi^{(0)}\delta_1\delta_2.$$

Since $k' \neq 1$,† we have $\xi\xi^{(0)}\delta_1\delta_2 = 1/(1-k')\,\xi\eta\zeta\tau$, and since $\xi\eta\zeta\tau = \xi_1\eta_1\zeta_1\tau_1 + \beta\xi^{(0)}\delta_1\delta_2$, where $\xi_1\eta_1\zeta_1\tau_1 = -k'\alpha\xi^{(0)}\delta_1\delta_2$, we have by theorem 20,

$$\xi\xi^{(0)}\delta_1\delta_2 = 1/(1-k')\,\xi_1\eta_1\zeta_1\tau_1 + 1/(1-k')\,\beta\xi^{(0)}\delta_1\delta_2.$$

That is,

$$\xi\xi^{(0)}\delta_1\delta_2 - 1/(1-k')\,\beta\xi^{(0)}\delta_1\delta_2 = 1/(1-k')\,\xi_1\eta_1\zeta_1\tau_1.$$

But $\xi_1\eta_1\zeta_1\tau_1 = -k'\alpha\xi^{(0)}\delta_1\delta_2$ and consequently by the theorems of § 2 and definitions 1 and 2 of the present section we have,

$$\xi\xi^{(0)}\delta_1\delta_2 - 1/(1-k')\,\beta\xi^{(0)}\delta_1\delta_2 = -k'/(1-k')\,\alpha\xi^{(0)}\delta_1\delta_2,$$

or if we put

$$a' = -k'/(1-k'), \quad b' = 1/(1-k'), \quad a' + b' = 1,$$

we obtain

$$\xi\xi^{(0)}\delta_1\delta_2 = a'\alpha\xi^{(0)}\delta_1\delta_2 + b'\beta\xi^{(0)}\delta_1\delta_2. \tag{4}$$

---

* We treat here essentially $\xi^{(0)} \neq \alpha, \beta, \xi$ and $\xi\xi^{(0)}\delta_1\delta_2 \neq 0$, $\alpha\xi^{(0)}\delta_1\delta_2 \neq 0$, $\beta\xi^{(0)}\delta_1\delta_2 \neq 0$.

† If $k' = 1$, then by (1), $\xi\beta\Delta'\Delta'' = \xi\alpha\Delta'\Delta''$. But by theorem 24, $\alpha\xi\Delta'\Delta'' + \xi\beta\Delta'\Delta'' = \alpha\beta\Delta'\Delta''$. Hence $\alpha\xi\Delta'\Delta'' + \xi\alpha\Delta'\Delta'' = \alpha\beta\Delta'\Delta''$; that is, $\alpha\beta\Delta'\Delta'' = 0$ or $\alpha = \beta$ by axioms 1, 2 and theorem 5 of Chapter V.

In a similar manner we derive the relations

$$\xi\,\xi^{(0)}\delta_2\delta_3 = a'\alpha\,\xi^{(0)}\delta_2\delta_3 + b'\beta\,\xi^{(0)}\delta_2\delta_3, \tag{5}$$

$$\xi\,\xi^{(0)}\delta_1\delta_3 = a'\alpha\,\xi^{(0)}\delta_1\delta_3 + b'\beta\,\xi^{(0)}\delta_1\delta_3. \tag{6}$$

Now by theorem 27, since $\xi_0\delta_1\delta_2\delta_3 = 0$ we have,

$$\begin{aligned}
\xi\,\xi^{(0)}\delta_2\delta_3 + \xi\,\delta_1\xi^{(0)}\delta_3 + \xi\,\delta_1\delta_2\xi^{(0)} &= \xi\,\delta_1\delta_2\delta_3,\\
\alpha\,\xi^{(0)}\delta_2\delta_3 + \alpha\,\delta_1\xi^{(0)}\delta_3 + \alpha\,\delta_1\delta_2\xi^{(0)} &= \alpha\,\delta_1\delta_2\delta_3,\\
\beta\,\xi^{(0)}\delta_2\delta_3 + \beta\,\delta_1\xi^{(0)}\delta_3 + \beta\,\delta_1\delta_2\xi^{(0)} &= \beta\,\delta_1\delta_2\delta_3,
\end{aligned}$$

and therefore, by using axiom G_{22} and an obvious extension of theorem 20,

$$\begin{aligned}
a'\alpha\,\xi^{(0)}\delta_2\delta_3 + a'\alpha\,\delta_1\xi^{(0)}\delta_3 + a'\alpha\,\delta_1\delta_2\xi^{(0)} &= a'\alpha\,\delta_1\delta_2\delta_3,\\
b'\beta\,\xi^{(0)}\delta_2\delta_3 + b'\beta\,\delta_1\xi^{(0)}\delta_3 + b'\beta\,\delta_1\delta_2\xi^{(0)} &= b'\beta\,\delta_1\delta_2\delta_3.
\end{aligned}$$

Hence by addition (cf. theorem 22) from (4), (5), (6),

$$\xi\,\delta_1\delta_2\delta_3 = a'\alpha\,\delta_1\delta_2\delta_3 + b'\beta\,\delta_1\delta_2\delta_3. \tag{7}$$

CASE II. The plane $\delta_1\delta_2\delta_3$ does not intersect* the line $\alpha\beta$.
Then we have †

$$\xi\,\delta_1\delta_2\delta_3 = \alpha\,\delta_1\delta_2\delta_3 = \beta\,\delta_1\delta_2\delta_3.$$

Now by Case I for three points $\delta_1', \delta_2', \delta_3'$ such that the plane $\delta_1'\delta_2'\delta_3'$ meets the line $\alpha\beta$ in a uniqe point,

$$\xi\,\delta_1'\delta_2'\delta_3' = a'\alpha\,\delta_1'\delta_2'\delta_3' + b'\beta\,\delta_1'\delta_2'\delta_3',$$

where $a' + b' = 1$ and a', b' correspond to elements of the class $\mathfrak{Q}$. Since $a' + b' = 1$ we may write

$$\xi\,\delta_1\delta_2\delta_3 = (a' + b')\,\alpha\,\delta_1\delta_2\delta_3,$$

and therefore, since a', b' correspond to elements of the class $\mathfrak{Q}$, by theorem 21,

$$\xi\,\delta_1\delta_2\delta_3 = a'\alpha\,\delta_1\delta_2\delta_3 + b'\alpha\,\delta_1\delta_2\delta_3.$$

But from the preceding, $\alpha\,\delta_1\delta_2\delta_3 = \beta\,\delta_1\delta_2\delta_3$, and hence by definition 2,

$$\xi\,\delta_1\delta_2\delta_3 = a'\alpha\,\delta_1\delta_2\delta_3 + b'\beta\,\delta_1\delta_2\delta_3.$$

CASE III. The plane $\delta_1\delta_2\delta_3$ passes through the line $\alpha\beta$.
Then

$$\xi\,\delta_1\delta_2\delta_3 = \alpha\,\delta_1\delta_2\delta_3 = \beta\,\delta_1\delta_2\delta_3 = 0.$$

---

* See the definitions under (I) of the present chapter.

† By the theorems of Chapter V and axiom G_{23} there exists a point θ such that, say, $\delta_1\delta_3\theta = 0$ and $\theta\,\delta_2\,\alpha\,\Delta = \theta\,\delta_2\,\beta\,\Delta$; putting $\Delta = \delta_3, \delta_1$ and using theorems 22, 24, we obtain by addition, $\alpha\,\delta_1\delta_2\delta_3 = \beta\,\delta_1\delta_2\delta_3$.

Therefore, by definition 2, for the two numbers a', b' such that $a' + b' = 1$ and a', b' correspond to elements of the class $\mathfrak{Q}$ (cf. Case I), we have

$$\xi\, \delta_1\, \delta_2\, \delta_3 = a'\, \alpha\, \delta_1\, \delta_2\, \delta_3 + b'\, \beta\, \delta_1\, \delta_2\, \delta_3 .$$

Suppose, secondly, that $\delta_1 \delta_2 \delta_3 = 0$; then, as in Case III,

$$\xi\, \delta_1\, \delta_2\, \delta_3 = \alpha\, \delta_1\, \delta_2\, \delta_3 = \beta\, \delta_1\, \delta_2\, \delta_3 ,$$

and we show similarly,

$$\xi\, \delta_1\, \delta_2\, \delta_3 = a'\, \alpha\, \delta_1\, \delta_2\, \delta_3 + b'\, \beta\, \delta_1\, \delta_2\, \delta_3 .$$

Therefore the theorem is verified.

Theorem 29.* If $\alpha \beta \gamma \neq 0$ and $\alpha \beta \gamma \xi = 0$, then $\xi \Delta_1 \Delta_2 \Delta_3 = a' \alpha \Delta_1 \Delta_2 \Delta_3 + b' \beta \Delta_1 \Delta_2 \Delta_3 + c' \gamma \Delta_1 \Delta_2 \Delta_3$, where $a' + b' + c' = 1$.

If $\xi = \alpha$ or β or γ, then by definition 2 the theorem is easily verified.

Let $\xi \neq \alpha, \beta, \gamma$. Suppose that ξ is collinear with $\alpha \beta$ or $\beta \gamma$ or $\gamma \alpha$, say, $\alpha \beta \xi = 0$. Then by theorem 28 and definition 2 we have

$$\xi\, \Delta_1 \Delta_2 \Delta_3 = a' \alpha\, \Delta_1 \Delta_2 \Delta_3 + b' \beta\, \Delta_1 \Delta_2 \Delta_3 + 0 \cdot \gamma\, \Delta_1 \Delta_2 \Delta_3 ,$$

and similarly if $\beta \gamma \xi = 0$ or $\gamma \alpha \xi = 0$. Let ξ be non-collinear with $\alpha \beta$ or $\beta \gamma$ or $\gamma \alpha$. Then there exists (see proof of theorem 26) a point η such that, say, $\beta \xi \eta = 0$ and $\alpha \gamma \eta = 0$. Hence, by theorem 28,

$$\begin{aligned} \xi\, \Delta_1 \Delta_2 \Delta_3 &= b \beta\, \Delta_1 \Delta_2 \Delta_3 + y \eta\, \Delta_1 \Delta_2 \Delta_3 , & b + y &= 1, \\ \eta\, \Delta_1 \Delta_2 \Delta_3 &= a \alpha\, \Delta_1 \Delta_2 \Delta_3 + c \gamma\, \Delta_1 \Delta_2 \Delta_3 , & a + c &= 1. \end{aligned}$$

By axiom G_{22}, definition 2, theorem 20 and the theorems of § 2, we find from the latter relations,

$$\xi\, \Delta_1 \Delta_2 \Delta_3 = y a \alpha\, \Delta_1 \Delta_2 \Delta_3 + b \beta\, \Delta_1 \Delta_2 \Delta_3 + c y \gamma\, \Delta_1 \Delta_2 \Delta_3 ,$$

or if we put

$$y a = a', \quad b = b', \quad c y = c',$$

then

$$a' + b' + c' = 1$$

and

$$\xi\, \Delta_1 \Delta_2 \Delta_3 = a' \alpha\, \Delta_1 \Delta_2 \Delta_3 + b' \beta\, \Delta_1 \Delta_2 \Delta_3 + c' \gamma\, \Delta_1 \Delta_2 \Delta_3 ,$$

which verifies the theorem.

Theorem 30. If $\alpha \beta \gamma \delta \neq 0$, then for any ξ, $\xi \Delta_1 \Delta_2 \Delta_3 = a' \alpha \Delta_1 \Delta_2 \Delta_3 + b' \beta \Delta_1 \Delta_2 \Delta_3 + c' \gamma \Delta_1 \Delta_2 \Delta_3 + d' \delta \Delta_1 \Delta_2 \Delta_3$, where $a' + b' + c' + d' = 1$.

---

* Compare Grassmann, *Werke*, Vol. I, 1, § 110.

If $\xi = \alpha$ or β or γ or δ, or if ξ is collinear with an edge of $\alpha\beta\gamma\delta$, or if ξ is coplanar with a face of $\alpha\beta\gamma\delta$, then by theorems 28 and 29 and definition 2, the theorem is easily verified.

Let now ξ be non-coplanar with each of the faces $\alpha\beta\gamma$, $\delta\beta\gamma$, $\alpha\delta\gamma$, $\alpha\beta\delta$ of $\alpha\beta\gamma\delta$. Then there exists (see proof of theorem 27) a point η such that, say, $\alpha\eta\xi = 0$ and $\beta\gamma\delta\eta = 0$, and therefore by theorem 28 and 29 we have

$$\xi\Delta_1\Delta_2\Delta_3 = a\alpha\Delta_1\Delta_2\Delta_3 + y\eta\Delta_1\Delta_2\Delta_3,$$
$$\eta\Delta_1\Delta_2\Delta_3 = b\beta\Delta_1\Delta_2\Delta_3 + c\gamma\Delta_1\Delta_2\Delta_3 + d\delta\Delta_1\Delta_2\Delta_3.$$

Hence by axiom G_{22}, definition 2, theorem 20 and the theorems of § 2, we find from the latter relations,

$$\xi\Delta_1\Delta_2\Delta_3 = a'\alpha\Delta_1\Delta_2\Delta_3 + b'\beta\Delta_1\Delta_2\Delta_3 + c'\gamma\Delta_1\Delta_2\Delta_3 + d'\delta\Delta_1\Delta_2\Delta_3,$$

where

$$a = a', \quad yb = b', \quad yc = c', \quad yd = d'$$

and

$$a' + b' + c' + d' = 1.$$